

CONNECTION WITH PARALLEL TOTALLY SKEW-SYMMETRIC TORSION ON ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

DIMITAR MEKEROV

Abstract

In the present work¹ we consider an almost complex manifold with Norden metric (i. e. a metric with respect to which the almost complex structure is an anti-isometry). On such a manifold we study a linear connection preserving the almost complex structure and the metric and having a totally skew-symmetric torsion tensor. We consider the case when the manifold admits a connection with parallel totally skew-symmetric torsion and the case when such connection has a Kähler curvature tensor. We get necessary and sufficient conditions for an isotropic Kähler manifold with Norden metric.

Key words: Norden metric, almost complex manifold, indefinite metric, linear connection, Bismut connection, KT connection, totally skew-symmetric torsion, parallel torsion.

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1. INTRODUCTION

There is a strong interest in the metric connections with totally skew-symmetric torsion tensor (3-form). These connections arise in a natural way in theoretical and mathematical physics. For example, such a connection is of particular interest in string theory [1]. In mathematics this connection was used by Bismut to prove the local index theorem for non-Kähler Hermitian manifolds [2]. A connection with totally skew-symmetric torsion tensor is called a KT connection by physicists, and among mathematicians this connection is known as a Bismut connection.

In the present work we continue the investigations from [3] for a connection ∇' with totally skew-symmetric torsion on non-Kähler quasi-Kähler manifolds with Norden metric. There are proved some necessary and sufficient conditions the curvature tensor of ∇' to be Kählerian. In the case when this tensor is Kählerian, some relations between its scalar curvature and the scalar curvatures of other curvature-like tensors are obtained. Moreover, conditions for isotropic Kähler manifolds with Norden metric are get.

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Now we consider the case when ∇' has a parallel torsion. We obtain a relation between the scalar curvatures for ∇' and the Levi-Civita connection ∇ . We establish that the manifold is isotropic Kählerian with Norden metric iff these curvatures are equal. We obtain a necessary and sufficient condition for ∇' with parallel torsion be with Kähler curvature tensor. Moreover we show that if ∇' has a parallel torsion and a Kähler curvature tensor, then the manifold is isotropic Kählerian.

2. PRELIMINARIES

Let (M, J, g) be a $2n$ -dimensional *almost complex manifold with Norden metric*, i. e.

$$J^2x = -x, \quad g(Jx, Jy) = -g(x, y),$$

for all differentiable vector fields x, y on M . The *associated metric* \tilde{g} of g on M , given by $\tilde{g}(x, y) = g(x, Jy)$, is a Norden metric, too. The signature of both metrics is necessarily (n, n) .

Further, x, y, z, w will stand for arbitrary differentiable vector fields on M (or vectors in the tangent space of M at an arbitrary point $p \in M$).

The Levi-Civita connection of g is denoted by ∇ . The tensor field F of type $(0, 3)$ on M is defined by

$$(2.1) \quad F(x, y, z) = g((\nabla_x J)y, z).$$

It has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz).$$

In [4], the considered manifolds are classified into eight classes with respect to F : $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_1 \oplus \mathcal{W}_2, \mathcal{W}_1 \oplus \mathcal{W}_3, \mathcal{W}_2 \oplus \mathcal{W}_3, \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. The class \mathcal{W}_0 of the *Kähler manifolds with Norden metric* is contained in each of the other seven classes. It is determined by the condition $F(x, y, z) = 0$, which is equivalent to $\nabla J = 0$. The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ is the class of all almost complex manifolds with Norden metric.

The condition

$$(2.2) \quad \mathfrak{S}_{x,y,z} F(x, y, z) = 0,$$

where $\mathfrak{S}_{x,y,z}$ is the cyclic sum over x, y, z , characterizes the class \mathcal{W}_3 of the *quasi-Kähler manifolds with Norden metric*. This is the only class among the basic classes $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ of manifolds with non-integrable almost complex structure J .

Let $\{e_i\}$ ($i = 1, 2, \dots, 2n$) be an arbitrary basis of the tangent space of M at a point $p \in M$. The components of the inverse matrix of g , with respect to this basis, are denoted by g^{ij} .

Following [5], the *square norm* $\|\nabla J\|^2$ of ∇J is defined in [6] by

$$(2.3) \quad \|\nabla J\|^2 = g^{ij} g^{ks} g((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_s),$$

where it is proven that

$$(2.4) \quad \|\nabla J\|^2 = -2g^{ij}g^{ks}g((\nabla_{e_i}J)e_k, (\nabla_{e_s}J)e_j).$$

There, the manifold with $\|\nabla J\|^2 = 0$ is called an *isotropic-Kähler manifold with Norden metric*. It is clear that every Kähler manifold with Norden metric is isotropic-Kähler, but the inverse implication is not always true.

Let R be the curvature tensor of ∇ , i. e. $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]}z$. The corresponding (0, 4)-tensor is determined by $R(x, y, z, w) = g(R(x, y)z, w)$. The Ricci tensor ρ and the scalar curvature τ with respect to ∇ are defined by

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \quad \tau = g^{ij}\rho(e_i, e_j).$$

A tensor L of type (0, 4) with the properties

$$(2.5) \quad L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$(2.6) \quad \mathfrak{S}_{x, y, z} L(x, y, z, w) = 0 \quad (\text{the first Bianchi identity})$$

is called a *curvature-like tensor*. Moreover, if the curvature-like tensor L has the property

$$(2.7) \quad L(x, y, Jz, Jw) = -L(x, y, z, w),$$

it is called a *Kähler tensor* [7].

Let ∇' be a linear connection with a tensor Q of the transformation $\nabla \rightarrow \nabla'$ and a torsion tensor T , i. e.

$$(2.8) \quad \nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].$$

The corresponding (0, 3)-tensors are defined by

$$(2.9) \quad Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z).$$

The symmetry of the Levi-Civita connection implies

$$(2.10) \quad T(x, y) = Q(x, y) - Q(y, x), \quad T(x, y) = -T(y, x).$$

A linear connection ∇' on an almost complex manifold with Norden metric (M, J, g) is called a *natural connection* if $\nabla'J = \nabla'g = 0$. The last conditions are equivalent to $\nabla'g = \nabla'\tilde{g} = 0$. If ∇' is a linear connection with a tensor Q of the transformation $\nabla \rightarrow \nabla'$ on an almost complex manifold with Norden metric, then it is a natural connection iff the following conditions are valid:

$$(2.11) \quad F(x, y, z) = Q(x, y, Jz) - Q(x, Jy, z),$$

$$(2.12) \quad Q(x, y, z) = -Q(x, z, y).$$

According to [8], we have

$$(2.13) \quad Q(x, y, z) = \frac{1}{2}\{T(x, y, z) - T(y, z, x) + T(z, x, y)\}.$$

Let ∇' be the natural connection with a totally skew-symmetric torsion tensor T on a non-Kähler manifold with Norden metric (M, J, g) . According to [3] we have

$$(2.14) \quad Q(x, y) = \frac{1}{4} \{ (\nabla_x J) Jy - (\nabla_{Jx} J) y - 2 (\nabla_y J) Jx \}.$$

Since ∇' has a totally skew-symmetric torsion tensor T , then

$$(2.15) \quad T(x, y, z) = -T(y, x, z) = -T(x, z, y) = -T(z, y, x).$$

From (2.13) and (2.15) it follows that for the tensor Q it is valid

$$(2.16) \quad Q(x, y, z) = \frac{1}{2} T(x, y, z).$$

3. CONNECTION WITH PARALLEL TOTALLY SKEW-SYMMETRIC TORSION

Let ∇' be the connection with totally skew-symmetric torsion tensor T on the quasi-Kähler manifold with Norden metric (M, J, g) .

Now we consider the case when ∇' has a parallel torsion, i. e. $\nabla' T = 0$.

It is known that the curvature tensors R' and R of ∇' and ∇ , respectively, satisfy

$$(3.1) \quad \begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) \\ &\quad + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \end{aligned}$$

Equality (2.16) implies $\nabla' Q = 0$ in the considered case. Then from the formula for covariant derivation with respect to ∇' it follows that

$$(3.2) \quad xQ(y, z, w) - Q(\nabla'_x y, z, w) - Q(y, \nabla'_x z, w) - Q(y, z, \nabla'_x w) = 0.$$

According to the first equality of (2.8) we have

$$(3.3) \quad \begin{aligned} Q(\nabla'_x y, z, w) &= Q(\nabla_x y, z, w) + Q(Q(x, y), z, w), \\ Q(y, \nabla'_x z, w) &= Q(y, \nabla_x z, w) + Q(y, Q(x, z), w), \\ Q(y, z, \nabla'_x w) &= Q(y, z, \nabla_x w) + Q(y, z, Q(x, w)). \end{aligned}$$

Combining (3.2), (3.3), the first equality of (2.9) and having in mind the formula for covariant derivation with respect to ∇ , we obtain

$$(3.4) \quad \begin{aligned} (\nabla_x Q)(y, z, w) &= Q(Q(x, y), z, w) \\ &\quad - g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)). \end{aligned}$$

From (3.4) and the first equality of (2.10) we have

$$(3.5) \quad \begin{aligned} (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) &= Q(T(x, y), z, w) \\ &\quad - 2g(Q(x, z), Q(y, w)) + 2g(Q(y, z), Q(x, w)). \end{aligned}$$

Because of (3.5), equality (3.1) can be rewritten as

$$(3.6) \quad \begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + Q(T(x, y), z, w) \\ &\quad - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)). \end{aligned}$$

Since $Q(e_i, e_j) = -Q(e_j, e_i)$ it follows that $g^{ij}Q(e_i, e_j) = 0$. Then, from (3.6) after contraction by $x = e_i$, $w = e_j$, we obtain the following equality for the Ricci tensor ρ' of ∇' :

$$(3.7) \quad \begin{aligned} \rho'(y, z) &= \rho(y, z) + 2g^{ij}g(Q(e_i, y), Q(z, e_j)) \\ &\quad - g^{ij}g(Q(e_i, z), Q(y, e_j)). \end{aligned}$$

Contracting by $y = e_k$, $z = e_s$ in (3.7), we get

$$(3.8) \quad \tau' = \tau + g^{ij}g^{ks}g(Q(e_i, e_k), Q(e_s, e_j)),$$

where τ' is the scalar curvature of ∇' .

By virtue of (3.8), (2.14), (2.3) and (2.4) we have

$$(3.9) \quad \tau' = \tau - \frac{1}{8} \|\nabla J\|^2.$$

Thus we arrive at the following

Theorem 3.1. *Let ∇' be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric (M, J, g) . Then for the Ricci tensor ρ' and the scalar curvature τ' of ∇' are valid (3.7) and (3.9), respectively. \square*

Equality (3.9) leads to the following

Corollary 3.2. *Let ∇' be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric (M, J, g) . Then the manifold (M, J, g) is isotropic Kählerian iff ∇' and ∇ have equal scalar curvatures. \square*

4. CONNECTION WITH PARALLEL TOTALLY SKEW-SYMMETRIC TORSION AND KÄHLER CURVATURE TENSOR

Let ∇' be a connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric (M, J, g) .

We will find conditions for the curvature tensor R' of ∇' to be Kählerian.

From (2.16), having in mind that Q is a 3-form, we have

$$\begin{aligned} Q(T(x, y), z, w) &= Q(z, w, T(x, y)) = g(Q(z, w), T(x, y)) \\ &= g(T(x, y), Q(z, w)) = 2g(Q(x, y), Q(z, w)). \end{aligned}$$

Then (3.6) obtains the form

$$(4.1) \quad \begin{aligned} R'(x, y, z, w) = & R(x, y, z, w) + 2g(Q(x, y), Q(z, w)) \\ & - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)). \end{aligned}$$

From (4.1), identities (2.5) and (2.7) for R' follow immediately. Therefore R' is a Kähler tensor iff the first Bianchi identity (2.6) for R' is satisfied. Since this identity is valid for R , then (4.1) implies that R' is Kählerian iff

$$\mathfrak{S}_{x,y,z} \{ 2g(Q(x, y), Q(z, w)) - g(Q(x, z), Q(y, w)) + g(Q(y, z), Q(x, w)) \} = 0.$$

Thus, using that Q is a skew-symmetric tensor, we arrive the following

Theorem 4.1. *Let ∇' be the connection with parallel totally skew-symmetric torsion on the quasi-Kähler manifold with Norden metric (M, J, g) . Then the curvature tensor for ∇' is a Kähler tensor iff*

$$(4.2) \quad \mathfrak{S}_{x,y,z} g(Q(x, y), Q(z, w)) = 0.$$

□

Because of the skew-symmetry of Q , (4.2) implies

$$g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) = -g(Q(x, y), Q(z, w)).$$

The last equality and (4.1) lead to the following

Corollary 4.2. *Let ∇' be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric (M, J, g) . Then*

$$(4.3) \quad R'(x, y, z, w) = R(x, y, z, w) + g(Q(x, y), Q(z, w)).$$

□

If R' is a Kähler tensor then $R'(x, y, Jz, Jw) = -R'(x, y, z, w)$, and because of (4.3) we have

$$(4.4) \quad \begin{aligned} R(x, y, Jz, Jw) + R(x, y, z, w) = & -g(Q(x, y), Q(Jz, Jw)) \\ & - g(Q(x, y), Q(z, w)). \end{aligned}$$

From (2.14) we get

$$Q(x, Jy) = JQ(x, y) - (\nabla_x J) y.$$

Then we have

$$Q(Jx, Jy) = -Q(x, y) - (\nabla_{Jx} J) y - (\nabla_y J) Jx$$

and consequently

$$\begin{aligned} g(Q(x, y), Q(Jz, Jw)) = & -g(Q(x, y), Q(z, w)) \\ & - g(Q(x, y), (\nabla_{Jz} J) w + (\nabla_w J) Jz). \end{aligned}$$

The last equality and (4.4) imply the following

Corollary 4.3. *Let ∇' be the connection with parallel totally skew-symmetric torsion and Kähler curvature tensor on the quasi-Kähler manifold with Norden metric (M, J, g) . Then*

$$(4.5) \quad R(x, y, Jz, Jw) + R(x, y, z, w) = g(Q(x, y), (\nabla_{Jz} J) w + (\nabla_w J) Jz).$$

□

Contracting by $x = e_i$, $w = e_j$ in (4.5), we obtain

$$g^{ij} R(e_i, y, Jz, Je_j) + \rho(y, z) = g^{ij} g(Q(e_i, y), (\nabla_{Jz} J) e_j + (\nabla_{e_j} J) Jz).$$

Then, after a contraction by $y = e_k$, $z = e_s$, it follows

$$(4.6) \quad \tau^{**} + \tau = g^{ij} g^{ks} g(Q(e_i, e_k), (\nabla_{Je_s} J) e_j + (\nabla_{e_j} J) Je_s),$$

where $\tau^{**} = g^{ij} g^{ks} R(e_i, e_k, Je_s, Je_j)$.

From (2.14), (2.3) and (2.4) we have

$$g^{ij} g^{ks} g(Q(e_i, e_k), (\nabla_{Je_s} J) e_j + (\nabla_{e_j} J) Je_s) = -\frac{1}{8} \|\nabla J\|^2.$$

Then (4.6) can be rewritten as

$$\tau^{**} + \tau = -\frac{1}{8} \|\nabla J\|^2.$$

On the other hand, according to [6], we have

$$\tau^{**} + \tau = -\frac{1}{2} \|\nabla J\|^2.$$

Then $\|\nabla J\|^2 = 0$ and therefore the following is valid.

Theorem 4.4. *Let (M, J, g) be a quasi-Kähler manifold with Norden metric which admit a connection with parallel totally skew-symmetric torsion and Kähler curvature tensor. Then (M, J, g) is a isotropic Kähler manifold with Norden metric.* □

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Dimitar Mekerov
Department of Geometry
Faculty of Mathematics and Informatics
Paisii Hilendarski University of Plovdiv
236 Bulgaria Blvd.
4003 Plovdiv, Bulgaria
e-mail: mircho@uni-plovdiv.bg